Introduction to Integral Calculus

Introduction

It is interesting to note that the beginnings of integral calculus actually predate differential calculus, although the latter is presented first in most text books. However in regards to formal, mature mathematical processes the differential calculus developed first. The first steps towards integral calculus actually began in ancient Greece. In the third century B.C., Aristotle became interested in areas defined by certain curves. He used rectangles to approximate these regions, and then used smaller and smaller rectangles, so that the approximation became better and better. You might note that this is not unlike some of the early methods trying to approximate the slope of a line that eventually led to differential calculus. He called this procedure the "method of exhaustion".

The famous mathematician Riemann would later generalize this procedure, using the concepts of limits that were developed for differential calculus. The process, often referred to as a “Riemann Sum”, is similar to Aristotle's rectangles, but the rectangles need not have a uniform thickness. Also, Riemann's method generalizes to higher dimensions, e.g. computing the volume bounded by a surface. There is an interesting Java applet on the web that illustrates how Riemann Sums work.

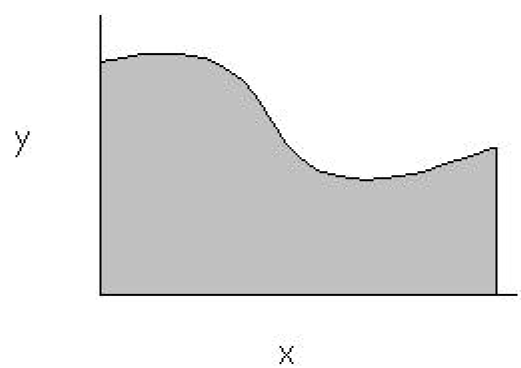
http://online.redwoods.edu/instruct/bwagner/applets/RiemannSums.html

Since Riemann Sums are not used today to calculate the area of an shape defined by a curve, the specific mathematics of them are not crucial to our discussion of integral calculus. There are, however, several interesting websites that do discuss the mathematics of the Riemann Sums:

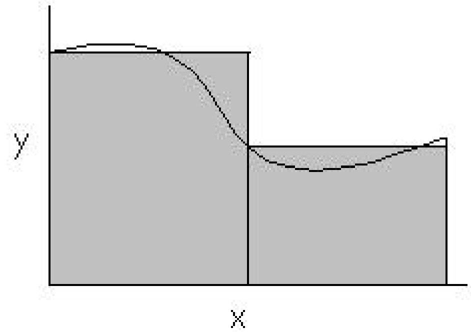
http://planetmath.org/encyclopedia/RiemannSum.html http://en.wikipedia.org/wiki/Riemann_sum

Differential calculus was primarily concerned with the slope of a line tangent to a curve at a given point. This was helpful in a variety of problems including computing instantaneous velocity and acceleration. Integral calculus is concerned with the area between that curve and the x axis. Calculating the area of a square, rectangle, triangle, and other regular polygons (or even a circle) is a trivial task of plugging in known measurements into formulas. If you wish to know the area of a right triangle you simply take ½ bh. If you want the area of a circle it is pi * r^2. Even more complex shapes such as octagons have formulas you can easily use.

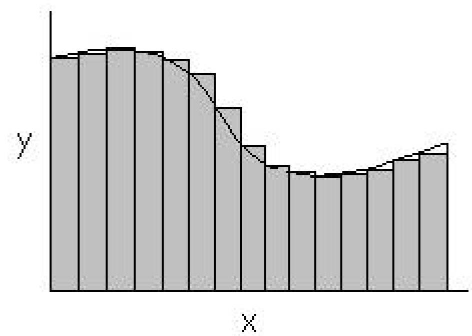
A serious problem arises when one wishes to calculate the area of an irregular curve. Such shapes cannot easily be plugged into a convenient formal and the area produced. Take the following function for example:



If the function were a horizontal line, the area would simply be x*y. But it is not a straight line, it is an irregular curve, making the answer significantly more difficult to compute. A good way to start approximating the area is to draw rectangles inside the area represented by the function. This simplifies the problem because finding the area of a rectangle is a trivial matter. This is, in fact, how Aristotle would have approached the problem. We might begin by doing this:



It is a trivial task to calculate the areas of both rectangles, then add them together. The resulting value would be a rough approximation of the area of the curve we are studying. Now we have a good approximation of the area under the curve. But you can see that our approximation is going to be significantly off. The answer to that is to make successively thinner rectangles. You might decide to try something like this:



As your rectangles get progressively narrower, your approximation will get better, but it will always be just an approximation, not the true area. It should also be noted that the more irregular the curve the more difficult it would be to get approximations via this method.

Integral calculus helps us find that area, and is in essence the opposite of differential calculus. Another term for integration is anti differentiation[1]. If $f(x)$ differentiates to $F(x)$ then, by definition, $F(x)$ integrates to give $f(x)$. We have been calling $F(x)$ the derivative of $f(x)$. We use the notation:

$$f(x) = F(x) \int \mathrm{d}x$$

The symbol $\int$ is the symbol for integration. When you differentiate an equation you get the slope. When you integrate you get the area between equation and the x-axis[1].

Put another way "The integral or anti-derivative of a function is another function such that the derivative of that function is equal to the original function. That is if G(x) is the anti-derivative of F(x), then the derivative of G(x) is equal to F(x). "[3]

To illustrate the concept of the anti derivative consider these simple examples:

Since the derivative of x^2+4 is 2x, an anti-derivative of 2x is x^2+4.

Since the derivative of x^2+10 is also 2x, another anti-derivative of 2x is x2+10.

Similarly, another anti-derivative of 2x is $x^2–20$.

Similarly, another anti-derivative of 2x is $x^2 + C$, where C is any constant (positive, negative, or zero) In fact: Every anti-derivative of 2x has the form $x^2 + C$, where C is constant. This is written in the following format:

$\int 2x\,dx = x^2 + C$

Here is how we read the formula:

$\int$	2x	dx	=	$x^2 + C$
The anti-derivative	of 2x,	with respect to x,	equals	$x^2 + C$

Many anti-derivatives follow similar patterns. The following table will examine a few of these:

Function	***Antiderivative***	***Formula***
x^n ($n \neq -1$)	$\dfrac{x^{n+1}}{n+1} + C$	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
Example: $\int 3x^{5.4}\,dx = \dfrac{3x^{6.4}}{6.4} + C$		
x^{-1}	$\ln\lvert x\rvert + C$	$\int x^{-1}\,dx = \ln\lvert x\rvert + C$
Example: $\int (5x^{-1} + 11x^{-3})\,dx = 5\ln\lvert x\rvert - \dfrac{11x^{-2}}{2} + C$		
k (k constant)	$kx + C$	$\int k\,dx = kx + C$
Example: $\int (5x^{-5.4} + 9)\,dx = -\dfrac{5x^{-4.4}}{4.4} + 9x + C$		
e^x	$e^x + C$	$\int e^x\,dx = e^x + C$
Example: $\int (3x^{5.4} + 9e^x - 4)\,dx = \dfrac{3x^{6.4}}{6.4} + 9e^x - 4x + C$		

Techniques of Integration

There are many techniques that can be applied to an individual integration problem.

Clearly one can memorize several of the commonly known integration formulas, some of which where shown in the preceding table. However these will not be applicable to all situations. In this section we will examine some additional techniques.

We will begin with substitution, a technique we also used in differential calculus. Some mathematicians regard substitution as the opposite of the chain rule used in differentiation. Lets look, step by step, at an example and its solution using substitution.

$\int 5x(2x^2+1)^{-3}\, dx$

to illustrate the main steps in changing a variable through substitution. First, we must decide what function to represent as u. Let us follow the advice in Example 2 on p. 444 in Calculus Applied to the Real World, or p. 942 inFinite Mathematics and Calculus Applied to the Real World.

Take u to be an expression that is being raised to a power.

Thus, we take

$u = 2x^2+1$.

We now follow a fairly mechanical step-by-step procedure:

Step	***Process***
Step 1: Calculate the derivative of u, and then solve for "dx."	$u = 2x^2+1$ $\frac{du}{dx} = 4x$ $dx = \frac{1}{4}\, du$

	$4x$
Step 2: Substitute the expression for u in the original integral, and also substitute for dx.	$\int 5x(2x^2+1)^{-3}\,dx = \int 5x(u)^{-3}\,\frac{1}{4x}\,du$
Step 3: Eliminate the variable x, leaving an integral in u only. **Often, as happens here, all the x's will cancel leaving an expression in u only.**	$\int 5x(u)^{-3}\,\frac{1}{4x}\,du = \int 5(u)^{-3}\,\frac{1}{4}\,du$
Step 4: Simplify the integrand. **Here, we can slip the constant 5/4 outside the integral sign.**	$\int 5(u)^{-3}\,\frac{1}{4}\,du = \frac{5}{4}\int u^{-3}\,du$
Step 5: Evaluate the simplified integral. **Important: Do not substitute back for u until after this step.**	$\frac{5}{4}\int u^{-3}\,du = \frac{5}{4}\,\frac{u^{-2}}{-2} + C$ $= -\frac{5u^{-2}}{8} + C$
Final Step: **Substitute back for u to obtain the result.**	$\int 5x(2x^2+1)^{-3}\,dx = -\frac{5(2x^2+1)^{-2}}{8} + C$

Integration by Parts

First consider this function:

$f(x) = x e^x$

This function is actually the product, $u(x)v(x)$, of the two functions, $u(x) = x$ and $v(x) = e^x$. Recalling techniques for finding derivatives should make a person consider the product rule. Applying the product rule you have

$d(x e^x)/dx = d(u(x)v(x))/dx = u(x)(dv/dx) + v(x)(du/dx) = x(d(e^x)/dx) + e^x (dx/dx)$

Recall that that the derivative of e^x is e^x, and also recall that $dx/dx = 1$. So we can rewrite the equation above as

$$\frac{d(x\ e^x)}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x\ e^x + e^x = (x + 1)\ e^x$$

Now we will take the indefinite integral of each expression:

$$\boxed{\int \frac{d(x\ e^x)}{dx}\,dx = \int \frac{d(uv)}{dx}\,dx = \int \left(u\frac{dv}{dx} + v\frac{du}{dx} \right) dx = \int (x\ e^x + e^x)\,dx}$$

Consider that the last two integrals are both integrals of sums. So we can rewrite this as

$$\boxed{\int \frac{d(x\ e^x)}{dx}\,dx = \int \frac{d(uv)}{dx}\,dx =}$$

$$\int u \frac{dv}{dx}\, dx + \int v \frac{du}{dx}\, dx = \int x\, e^{x}\, dx + \int$$

Recall that the Fundamental Theorem of Calculus tells you that taking a derivative and taking an indefinite integral are inverse operations of each other. So by taking the indefinite integral of the derivative of something, you end up getting back that same something (plus an undetermined constant).

$$x\, e^{x} + C = uv + C =$$

$$\int u \frac{dv}{dx}\, dx + \int v \frac{du}{dx}\, dx = \int x\, e^{x}\, dx + \int \qquad e^{x}\, dx$$

If you consider at the last term of the last expression, you will see that it is relatively easy

$$x\, e^{x} + C = uv + C =$$

$$\int u \frac{dv}{dx}\, dx + \int v \frac{du}{dx}\, dx = \int x\, e^{x}\, dx + e^{x} + C' \qquad \text{eq. 11.4-3d}$$

$$\textbf{dx} \qquad\qquad \textbf{dx}$$

(where C' is another undetermined constant).

Now equate just the first and last expressions of this equation and subtract $e^x + C'$ from both sides:

$$x\, e^x - e^x + C'' = \int x\, e^x\, dx$$

where C" is the difference of the two constants

Foot Notes

1. The Integral, 2004, http://www.maths.abdn.ac.uk/~igc/tch/ma1002/int/node5.html

2. Quick Calculus 2nd Edition, Kleppner and Ramsey, John Wiley and Sons Publishing 1985

3. Techniques of Basic Integration,2004, http://students.uww.edu/muellerbt15/BI.htm.

4. The Standard Deviants (Video Calculus Lesson), 1998 Cerebellum Corporation.

5. Calculus Net, 2004, http://www.calculus.net/

6. Planet Math, 2004, http://planetmath.org/encyclopedia/ChainRule.html