A Basic Overview of Introductory Calculus
by
Chuck Easttom

Contents

Introduction

The development of calculus is usually attributed to Isaac Newton and Gottfried Wilhelm Liebeniz in the 17th century[1]. Both mathematicians developed the concepts and

1

methods of modern calculus at the same time, unbeknownst to each other. However, like almost all scientific advances, the development of calculus was not done in an intellectual vacuum. Many advances in mathematics over the preceding years had set the stage for this development. Scientists like Galileo and the Italian mathematician Cavalieri, had been working on various approximations for measuring volumes and for working with curves, both of which are fundamental to calculus. Cavalieri's method was to use what he termed 'indivisibles'[2]. He essentially assumed a curve to be merely the sum of its points. If one assumes an infinite number of such points, one can get an increasingly accurate description of the curve. This concept is not that far from the modern concept of differentiation using the slope of the tangent line. The tangent line is, in essence, studying the behavior of the curve at the point where that tangent line intersects the curve.

The famous mathematician Rene Descartes had also been using algebraic methods to study curves. A tremendous step towards modern calculus was that of the mathematician Gilles Persone de Roberval, who was a professor at the Collège Royale for forty years[3]. Roberval considered the problem of instantaneous motion along a curve, a problem familiar to all who have taken introductory calculus courses. He thought that if you took any point on the curve, and summed the vectors making that motion, that sum would be the tangent and you would have a description of the instantaneous motion at that point. This was an important step forward, since students of modern calculus spend a great deal of time studying the slope of the tangent at a given point, other wise known as the derivative.

Pierre Fermat, one of the greatest personas in the history of mathematics, had a method for discovering tangents on a curve that was very close to what Liebeniz and

Newton actually used in their calculus. The study of tangents to a curve is a central theme in the study of limits and instantaneous velocity.

It is important for the student of calculus to understand the many contributions made by various mathematicians throughout several centuries. The work of these mathematicians laid the foundation for what would become modern calculus. However it was Newton and Liebeniz who took these ideas and formulated the modern calculus that we use today. You will find that calculus has applications in a great many areas including physics, engineering, and statistics.

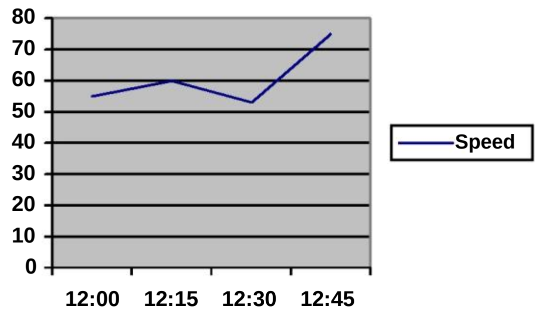
Limits

The limit is a rather simple mathematical concept, but one which some students have difficulty grasping the purpose for. Consider the problem of velocity. One need only use elementary algebra to understand the average velocity over a period of time. A formula such as

$(d2 - d1)/(t2 - t1)= V$ will suffice.

If you let d2 and t2 be the final location and the final time, and d1 and t1 be the starting location and time, we can take the change in distance divided by the change in time and get an average velocity. The problem arises if the velocity did not remain constant throughout the time frame involved. The graph of such a journey would then have various curves indicating acceleration or deceleration in velocity. Like the graph shown here.

3



How does one compute the instantaneous velocity at a given moment in time? The answer brings our discussion back to Roboval's consideration of points on a curve. If you consider the line tangent to a point of interest, then the slope of that line will be the instantaneous velocity at that point in time[5]. You can consider what the velocity is at any point b, as point b moves closer to the point of interest. You can become infinitely close without actually reaching that point if you so desire.

This brings us to the concept of limits. When you consider a limit you are considering the behavior of a given function as it approaches closer and closer to a given point. Consider this simple equation:

$$n/n+1$$

You can now consider what happens as n approaches any number you wish. For example as n gets closer to 2, then n/n+1 gets closer to 1. So the limit of n/n+1 as n approaches 2 is 1. The basic algebra involved here is quite obvious.

However what occurs in situations where you can never actually reach a particular value? For example consider:

$$n/n-1$$

4

What happens as n approaches 1? If n ever actually reaches 1 then we have division by zero which is impossible, so we cannot consider what happens at n = 1. However we can get infinitely close to 1. We can get an intuitive feel for what a limit is by plugging in numbers for n that are close to 1 without actually putting 1 into the equation. Introductory calculus courses often introduce limits by showing a chart, like the one shown here:

N	**N/n-1**
2	2
1.5	3
1.25	5
1.175	6.714
1.1	11
1.005	201

It should be obvious that as n approaches 1.0 that n/n-1 will grow larger. In fact as n becomes infinitely close to 1.0, n/n-1 approaches infinity. So we sat that the limit as n -> 1 is infinity. This should also make intuitive sense. Recall from elementary arithmetic that division by zero is undefined. Consider that if you divide a number by 1, you are dividing it into a single segment, itself. If you divide by 2 you are creating 2 segments. But if you divide by zero, then it is not divided at all, thus you have infinity.

Lets examine another example to illustrate the concept of a limit.

Lim (n +1)/1 as n ->∞

The answer to this is ∞. This can be shown several ways, but the chart previously used is probably the easiest to understand.

5

N	**(n +1)/1**
1	2
10	11
100	101
10000	10001

The function (n + 1)/1 will always yield a number that is one greater than n. So as n approaches infinity, so will (n +1)/1.

A limit can be informally defined as the behavior of a function as its independent variable approaches infinitely close to a given value[6].

Intuitive Definition. Let **y** = **f(x)** be a function. Suppose that **a** and **L** are numbers such that

- whenever **x** is close to **a** but not equal to **a**, **f(x)** is close to **L**;
- as **x** gets closer and closer to **a** but not equal to **a**, **f(x)** gets closer and closer to **L**; and
- suppose that **f(x)** can be made as close as we want to **L** by making **x** close to **a** but not equal to **a**.

Then we say that **the limit of f(x) as x approaches a is L** and we write

$$\lim_{x \to a} f(x) = L$$

I prefer a simpler definition. A limit is an examination of the output of a function as the input approaches some particular value. This is particularly important with functions where our input can never reach the value we wish to study. Consider the function N/n-1.

6

We can never examine this in the case that n=1 because the value is undefined. However we can examine the behavior of the function as n approaches 1. We can get as close to one as we like. We can use n = 1.000001 or n = 1.0000000000001. We can get infinitely close to one without ever reaching 1. And that is essentially what a limit is. It is the study of a function as it approaches infinitely close to some value, but may or may not ever actually reach that value.

This gives us a basic concept of what a limit is, it is now important for us to examine a variety of examples to ensure that we have an understanding of limits that is sufficient for us to move forward into the study of derivatives.

Example 1: lim x ->1 of $(x^2 – 4)/(X+2)$

$= (1 – 4)/(1+2)$

$= -3/3 = -1$

In this case simple substitution was adequate for finding the limit of this function at the point given. However what if we consider the limit of the same function but at a different point:

Example 2: lim x->-2 of $(x^2 – 4)/(X+2)$

You can see that substitution will not work. You will get division by zero. So in this case we use the basic algebraic technique of factoring.

$= ((x+2)(X-2))/ (x+2)$

the (x+2) factors out leaving us with x-2 we can now substituted -2 for x and get a limit of –4.

7

In many cases you can use factoring in order to render a function into a form that is suitable for substitution. However factoring will not work in all circumstances. Therefore we need other methods:

Example 3: lim x ->4 of (√x -2)/(x –4) clearly substitution will not work since that takes us to 0/0. And just as clearly it is not possible to factor √x. Well there is yet another algebraic technique that can be applied here. You can multiply both sides of the equation by the conjugate of the element that has the √x in it. The conjugate is essentially the same equation with the sign reversed. So in this case the conjugate would be √x + 2 , So both numerator and denominator are both multiplied by √x +2. That will effectively remove the √ item since a √x multiplied by √x yields x. So we now have:

(x –4)/((x-4) √x +2)

which simplifies to 1/(√x + 2) and now we can substituted 4 in giving us

1/(√4 +2) or ¼

These examples should provide a basic overview of what a limit is and how to compute one. They also demonstrate some common techniques for finding a limit in a variety of circumstances. However thus far we have only discussed the general limit. One can also consider the limit of a function approaching from either side. Also referred to as left hand and right hand limits.

A left hand limit is denoted by lim x ->a^- which simply means the limit as x approaches a from the left. The right hand limit is denoted in a similar manner. It is shown by lim x ->a^+ which means the limit as x approaches a from the right.

Derivatives

8

A derivative is simply a rate of change. In graphing terms it is the slope of a tangent line to a curve at a given point. Mathematically a derivative is really just a special case of the limit. It is one used widely in physics and engineering. The derivative is just the limit of the change in y divided by the change in x, as y approaches 0. Here is that put in more formal format.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Let us consider, briefly the origin of this formal definition of a limit. Given that a derivative is the slope of a tangent line to a point on a curve, then the tangent is a straight line that can be defined by coordinates (x1, y1) and (x2, y2). It therefore follows that the slope of that line can be defined as (y2-y1)/(x2-x1). However we only have one point, the single point at which the tangent line touches the curve. We have x1, and y1, but we do not have x2, y2. What we need is the change in x and y. So one can substituted the delta x symbol for that change, or as many textbooks do, you can substitute h to represent the change in X. So since y and f(x) are the same thing, we now have the points (x,f(x)) and (x+h), f(x+h). Remember that the slope of any line is (y2 – y1)/(x2 – x1). This gives us the slope of this tangent line as (f(x + h) – f(x)) / ((x+h) – x). Since the x and –x in the denominator cancel each other out we are left with (f(x +h) – f(x))/ h as the slope of the line. This is the exact same formula as the one shown in the previous figure, except that h is used rather than the delta x symbol[7]. So you can see that the derivative is the rate of change of a function. As we will see later this concept has a wide range of applications including economics, statistics, physics, chemistry and more.

9

Depending on the context there exist multiple symbols to denote the derivative. The most common are listed here:

dy/dx Note that students new to calculus may think of this as a ratio dy over dx, that is not the case. If that where the case then the d’s would cancel and you would have y/x. It is instead, simply a symbol for differentiation.

y’

D note that the capital D is not encountered as often as the first two. The first two are by far the most common symbols for the derivative.

Next we will do a few basic examples which should illustrate the basics of differentiation:

Example 1: y’ x^2

Recall the formal definition/format of a limit is:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Therefore we can simply plug x^2 into this

format $=(x + h)^2 – x^2 / h$

$= ((x + h) (x + h) - x^2)/h$

$= (x^2 + 2xh + h^2 - x^2)/h$

$= (2 xh + h^2)/h$

$= 2x + h$

And as x approaches zero the change in x will become zero leaving us

with 2x

10

This can be generalized by saying that for any function x^n the derivative is nx^{n-1}. If x has a coefficient a, then you can say that for any function ax^n the derivative is nax^{n-1}.

Example 2: y' $3x^2 + 7x + 2$

Recall the definition of a derivative and plug this function into that formula:

f(x + h) becomes $3(x + h)^2 + 7(x + h) + 2$

$= 3((x+h)(x + h)) + 7x + 7h + 2$

$= 3(x^2 + 2xh + h^2) + 7x + 7h + 2$

$= 3x^2 + 6xh + 3h^2 + 7x + 7h + 2$

Don't forget that this is just f(x + h) we must now subtract f(x)

$= ((3x^2 + 6xh + 3h^2 + 7x + 7h + 2) – (3x^2 + 7x + 2)) / h$

$= (6xh + 3h^2 + 7h)/h$

$= (6x + 3h + 7)$

And now if you substitute in 0 for h you

get $= 6x + 7$

Example 3: y' of $y = x^4 + 8x^3$

In this case y is actually the addition of 2 functions. This can be rewritten in the more general form of y =u(x) + v(x). The derivative of a function that is the addition of two other functions is simply the derivative of each added. This means that y' of u(x) + v(x) is y' u(x) + y' v(x).With the function 4x4 + 8x3 you could certainly plug this into the formal format which we have used to this point. However this would lead to some rather convoluted and unnecessary algebra. Instead we apply two rules:

Rule 1: for any function y = u(x) + v(x) , y' = y'u(x) +

y'v(x) Rule 2: for any function ax^n, $y' = nax^{n-1}$

11

This means the derivative of x4 + 8x3 is simply the two derivatives added. Rule two tells us that y' x^4 is $4x^3$ and y' $8x^3$ is $24x^2$. So the derivative of $x^4 + 8x^3$ is $4x^3 + 24x^2$.

Now we can test that using example 2 and applying the same rules, with one additional rule. Recall that function was $3x^2 + 7x + 2$ and y' was 6x + 7. Let us apply what we know. This derivative could be found using the derivative of $3x^2$ plus the derivative of 7x plus the derivative of 2. Using the rule regarding the derivatives of functions with exponents the derivative of $3x^2$ becomes 6x. You can look at 7x as $7x^1$ which would give us a derivative of 7. The derivative of 2, as with any constant is 0. So we end up with y' $x^2 + 7x + 2$ once again equals 6x + 7.

Example 4: In this example we will introduce a new rule, this is the product rule. The product rule states that y'(uv) is uv' + vu'. In other words the product of 2 functions multiplied together is the quantity of the first function multiplied by the derivative of the second function and the second function multiplied by the derivative of the first function.

With that in mind let us consider the problem $y = (x^5 + 7)(x^3 + 17x)$. Using the product rule this means that y' $(x^5 + 7)(x^3 + 17x)$ will be

y'$(x^5 + 7)(x^3 + 17x)$ + y'$(x^3 + 17x)$ $(x^5 + 7)$

$= 5x^4(x^3 + 17x) + (3x^2 + 17)(x^5 + 7)$

Note some calculus books stop here without simplifying and grouping. $=(5x^7 + 85x^4) + (x^7 + 17x^5 + 21x^2 + 119)$

$= 6x^7 + 17x^5 + 85x^4 + 21x^2 + 119$

Since 17 and 119 are prime there is no way to further simplify this.

12

Example 5: In this example we will be applying the quotient rule. This rule is significantly more complex than the product rule. It is shown in this figure from the Visual Calculus webite[9]:

$$(f / g)'(x) = \frac{g(x)\, f'(x) - f(x)\, g'(x)}{[g(x)]^2}$$

Essentially this rule states that to find the derivative of a quotient u/v you take the derivative of u multiplied by v minus the derivative of v multiplied by u. So far this looks much like the product rule only you are subtracting rather than adding. However you now take that entire quantity and divide it by v^2. Let us consider this problem:

$y'\ (1 + x)/x^2$

Using previous rules we know that the derivative of (1 + x) must be 1. And the derivative of x^2 is 2x. So we have:

$(x^2\ (y'\ (1 + x) - (1 +x)(y'x^2))/$

$x^4 =(x^2 - (1+x)\ 2x)/ x^4$

$= (x^2/ x^4\ \ - 2x/ x^{4)}\ \ (1+x)$

$= (1/ x^2\ \ - \ 2/ x^3)\ (1+x)$

Example 6: In this example we will consider the derivative of a function of a function, or y' of f(g(x)). In this case we apply a rule known as the chain rule. This rule is shown in the following image from Planet Math[10]

$$(f \circ g)'(x) = f'(g(x))\, g'(x)$$

So let us consider y' of $(x + x^2)^2$ Now the chain rule tells us that we need to take the derivative of g of x and multiply it times the derivative of x. In this case x is $(x +x^2)$

f’ $g(x + x^2)$ is $(x + x^2)^2$ or $2(x + x^2)$ which is $2x + 2x^2$ and g’(x) is $1 + 2x$. So we now have

13

$(2x + 2x^2)(1 + 2x)$

$= 2x + 2x^2 + 4x^2 +$

$4x^3 = 4x^3 + 6x^2 + 2x$

or $2x(2x^2 + 3x + 1)$

Example 6: The chain rule is such an important part of the study of differential calculus that one more example is in order. Consider $(2x + 7x^2)^{-2}$

Using the chain rule we get $-2(2x + 7x)^{-3}(2 + 14x)$

General Rules for derivatives:

There are general rules for finding the derivatives of well-known functions. The past several pages have introduced many of them. Most calculus students will want to commit these to memory. Some of the more commonly used short cut rules are here:

The derivative of a constant is always 0

The derivative of ax^n is always nax^{n-1}

Therefore y' of $2x^3$ is $6x^2$, y' of $5x^2$ is 10x.

The derivative of sin x is cos x

The derivative of cos x is – sin x

The derivative of ln x is 1/x The

derivative of u + v is u' + v'

The derivative of uv is uv' + u'v

Second Order Derivatives

One can take a second (or even third, fourth, etc.) derivative of a function. This is literally done by taking the derivative of the derivative. Consider these examples

Example 1: $2x^3 - x^2 + x - 7$

14

y’ is $6x^2 - 2x + 1$ -First derivative

y’’ is 12 x –2 -Second derivative

y’’ is 12 -Third derivative

Practical Applications

As was stated earlier in this paper, calculus has a number of practical applications. We will look at just a few here. It should be noted that a comprehensive treatment of the practical applications of differential calculus would occupy several volumes.

Example 1 – Velocity: Recall from our discussion of limits that average velocity is found by simply dividing the distance moved by the time taken to move that distance.

$V_{average} = d/t$

Position at any given time is simply determined by the time in question. If it is 15 minutes since the start of the journey that time will indicate our position.

However instantaneous velocity is actually the derivative of the position function or $V_{instantaneous} = f'(t)$ or ds/dt

Acceleration is actually the measure of the change in velocity. This means it is the derivative of velocity, making it the second order derivative of the position function. Acceleration = $f''(t)$

Example 2 - The Free Fall Equation: Let us now apply some of the general concepts from Example 1, to a specific situation. The free fall equation considers the descent of an object due to the pull of gravity. Friction and wind resistance are ignored.

$s = 1/2gt^2$

s is the position, t is the time and g is the acceleration due to gravity and can be expressed as either $9.8m/s^2$ or $32ft/s^2$.

15

This equation gives us the position of the object as it falls. Now recall from example 1 that velocity is the derivative of the position function. So velocity would be

y’ $1/2gt^2$ So we apply some basic rules of derivatives and we know that the velocity is 2(1/2gt)

So if it is the first second of free fall then the velocity

is 2(1/2(9.8)1)

or 9.8 m/s

At two seconds of free fall, the velocity would

be 2(1/2(9.8)2)

or 19.6 m/s

So you can see that differential calculus has clear applications to physics.

16

Foot Notes

1. The History of Calculus, 2004, http://www.geocities.com/CapitolHill/5760/history.htm

2. The History of Calculus and the Development of Computer Algebra Systems, 2004, http://www.math.wpi.edu/IQP/BVCalcHist/calc1.html#_Toc407004350

3. History of the Differential from the 17th Century, 2004, http://www.math.wpi.edu/IQP/BVCalcHist/calc2.html

4. The Garden of Archimedes, 2004, http://www.math.unifi.it/archimede/archimede_inglese/mostra_calcolo/guida/node7.html

5. Quick Calculus 2nd Edition, Kleppner and Ramsey, John Wiley and Sons Publishing 1985

6. Numerical Introduction to Limits, http://archives.math.utk.edu/visual.calculus/1/limits.16/, 1995.

7. The Standard Deviants (Video Calculus Lesson), 1998 Cerebellum Corporation.

8. Calculus Net, 2004, http://www.calculus.net/

9. Visual Calculus, 2004, http://archives.math.utk.edu/visual.calculus/2/quotient_rule.4/

10. Planet Math, 2004, http://planetmath.org/encyclopedia/ChainRule.html

17